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NOTES ON THE RADICAL AXIS AND THE QUADRILATERAL.

By WILLIAM GALLATLY, Boscombe, England.

I. THE RADICAL AXIS.

1. If d be the distance of a point P from the center of a circle X of radius x, then $\pm(d^2-x^2)$ is called the power of P with respect to X. ABC being the triangle of reference, let ρ_1^2, ρ_2^2, ρ_3^2 be the powers of A, B, C, and let the circle X have the equation

$$a\beta\gamma+b\gamma\alpha+c\alpha\beta+k(a\alpha+b\beta+c\gamma)(l\alpha+m\beta+n\gamma)=0.$$

Now if $S=0$ be any conic, and if S' be the value of S when the coordinates of P, Cartesian or trilinear, are substituted for the general coordinates, it is known that S' is proportional to the power of P. Hence, for the point A, $\lambda\rho_1^2=k.2\triangle.l.2\triangle/a$; for A is $(2\triangle/a, 0, 0)$. Hence $kl=\lambda/4\triangle^2.a\rho_1^2$.

The radical axis of the circles X and ABC is thus

$$a\rho_1^2.\alpha+b\rho_2^2.\beta+c\rho_3^2.\gamma=0.$$

And the circle X takes the form

$$a\beta\gamma+b\gamma\alpha+c\alpha\beta+\frac{\lambda}{4\triangle^2}.(a\alpha+b\beta+c\gamma)(a\rho_1^2.\alpha+b\rho_2^2.\beta+c\rho_3^2.\gamma)=0.$$

2. We now proceed to find the radical axis of the circle ABC, and (1) the in-circle; (2) the nine point circle; (3) the polar circle; (4) the circle on Hb (H being the orthocenter).

For (1), $\rho_1^2=(s-a)^2$; so that the radical axis is $a(s-a)^2.\alpha+....=0$.

For (2), N being the nine point center,

$$\rho_1^2=AN^2-Nd^2=Ad^2+2.Ad.dN.\cos(B-C)$$
$$=R^2\cos A[\cos A+\cos(B-C)]=2R^2\cos A\ \sin B\ \sin C.$$

Hence $a\rho_1^2$ varies as $\cos A$. The radical axis is $\cos A.\alpha+....=0$.

For (3). The polar circle has center H, and $x^2=-4R^2\cos A\ \cos B\ \cos C$. Therefore, $\rho_1^2=AH^2-k^2=4R^2\cos^2 A+4R^2\cos A\ \cos B\ \cos C=4R^2\ \cos A\ \sin B \times \sin C$. Hence $a\rho_1^2$ varies as $\cos A$, and the radical axis is $\cos A.\alpha+....=0$.

For (4). Let ω be the mid-point of GH.

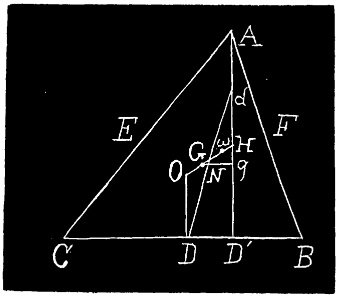
$$2\rho_1^2=2(A\omega^2-\omega H^2)=2(A\omega^2+\omega H^2)-Hb^2=Ab^2+AH^2-HG^2$$
$$=2AG.AH.\cos GAH=2AH.Ag=2.AH.\tfrac{2}{3}.AD'$$
$$=4R\ \cos A.\tfrac{2}{3}.2R\ \sin B\ \sin C.$$

Hence $a\rho_1^2$ varies as $\cos A$, and the radical axis is $\cos A.\alpha + \dots = 0$.

3. To determine the radical axis of the nine point circle, and (1) circum-circle, (2) in-circle.

It is now convenient to take the mid-point triangle DEF *as triangle of reference.*

If $(\alpha'\beta'\gamma')$ and $(\alpha\beta\gamma)$ be the old and new coordinates of a point, then $\alpha' + \alpha = \frac{1}{2}.AD' = 2\triangle/a$; all symbols now referring to DEF, so that $a\alpha' = b\beta + c\gamma$; and $l\alpha + m\beta + n\gamma = 0$ becomes



$$a\alpha\left(\frac{m}{b} + \frac{n}{c}\right) + \dots = 0.$$

Hence $\cos A.\alpha + \dots = 0$, becomes $a^3\alpha + \dots = 0$.

For (1), $x = 2R$; $d = OD = 2R\cos A$.

Hence $\rho_1^2 = 4R^2(1 - \cos^2 A)$. Therefore $a\rho_1^2$ varies as a^3, so that the radical axis is $a^3.\alpha + \dots = 0$, agreeing with Art. 2 (1).

For (2), $\rho_1^2 = (b-c)^2$; so that the radical axis is $a(b-c)^2.\alpha + b(c-a)^2.\beta + c(a-b)^2.\gamma = 0$. But this is the tangent to $a/\alpha + \dots = 0$ (the nine point circle) at

$$\left(\frac{1}{b-c},\ \frac{1}{c-a},\ \frac{1}{a-b}\right).$$

Hence (Feuerbach's Theorem), the nine point and in-circles touch; the point of contact being

$$\left(\frac{1}{b-c},\ \frac{1}{c-a},\ \frac{1}{a-b}\right).$$

II. The Quadrilateral.

1. Let the triangle ABC be cut by the straight line

$$\frac{a\alpha}{l} + \frac{b\beta}{m} + \frac{c\gamma}{n} = 0$$

in the points A', B', C'. Bisect AA', BB', CC' in A_1, B_1, C_1.

For A_1, $\frac{b\beta}{m} + \frac{c\gamma}{n} = 0$, and $b\beta + c\gamma = a\alpha$.

$\therefore \frac{b\beta}{m} = \frac{c\gamma}{-n} = \frac{a\alpha}{m-n} = \frac{\triangle}{m-n}$, with similar expressions for B'. Hence

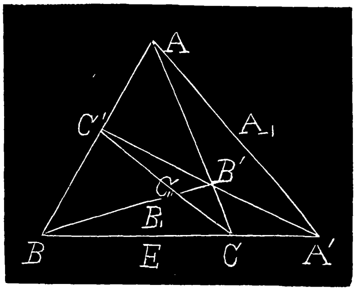
$$A'B' \text{ is } (m+n-l)a\alpha + (n+l-m)b\beta + (l+m-n)c\gamma = 0,$$

the symmetry of the equation showing that C' is also on this line.

2. If $BCB'C'$ is circumscriptible to a circle, then the mid-point line passes through the center. The in-circle of ABC is $\sqrt{}[a(s-a).\alpha+....=0$.

If $A'B'C'$ is a tangent, then $\dfrac{a\,(s-a)}{a/l}+....=0$, or $(m+n-l)a+....=0$. Hence the centre (1, 1, 1) lies on the mid-point line.

3. The orthocenters of the four triangles ABC, $AB'C'$, $BC'A'$, $CA'B'$, are collinear.



The perpendiculars from B' on AB, and C' on AC, are found to be

$$na.\cos A.\alpha+(lc-na\cos B)\beta+lc\cos A.\gamma=0.$$

$$ma.\cos A.\alpha+lb\cos A.\beta+(lb-ma\cos c)\gamma=0.$$

By subtraction we obtain the symmetrical equation

$$(m-n)\cos A.\alpha+(n-l)\cos B.\beta+(l-m)\cos C.\gamma=0,$$

which proves the proposition.

4. To find the equations of the circles $AB'C'$, $BC'A'$, $CA'B'$. Let the equation of $AB'C'$ be

$$a\beta\gamma+b\gamma\alpha+c\alpha\beta=k(a\alpha+b\beta+c\gamma).(m'\beta+n'\gamma)\ \ (l'\text{ obviously zero}).$$

At B', $\dfrac{a\alpha}{l}+\dfrac{c\gamma}{n}=0$; $\beta=0$. Hence, $\dfrac{a\alpha}{-l}=\dfrac{c\gamma}{n}=\dfrac{2\,\triangle}{n-l}$; and $b\gamma\alpha=k.2\triangle.n'\gamma$.

Hence, $kn'=\dfrac{b}{a}.\dfrac{-l}{n-l}$; so $km'=\dfrac{c}{a}.\dfrac{l}{l-m}$.

Hence, the equation of $AB'C'$ is

$$a\beta\gamma+b\gamma\alpha+c\alpha\beta=\frac{l}{a}(a\alpha+b\beta+c\gamma)\left(\frac{c}{l-m}.\beta-\frac{b}{n-l}.\gamma\right);$$

so that, if the circle $AB'C'$ cuts the circle ABC in D, the equation to AD is

$$\frac{c}{l-m}.\beta-\frac{b}{n-l}.\gamma=0,\text{ or }\beta\div\frac{b}{n-l}=\gamma\div\frac{c}{l-m}.$$

It follows that the circles $AB'C'$, $BC'A'$, $CA'B$ cut the circle ABC in the same point D, whose coordinates are $\left(\dfrac{a}{m-n},\ \dfrac{b}{n-l},\ \dfrac{c}{l-m}\right)$.

5. The condition that $BCB'C'$ may be cyclic, or that $B'C'$ may be antiparallel to BC. In this case it can be at once proved geometrically that D lies on AA' or $\frac{b\beta}{m}+\frac{c\gamma}{n}=0$. Hence the required condition is

$$\frac{b^2}{m(m-n)}+\frac{c^2}{n(l-m)}=0.$$

6. The Simson Line of D.

The Simson line of $(\alpha'\beta'\gamma')$ is known to be

$$\frac{a^2}{\alpha'^2}\cdot\frac{a}{\beta'\cos C-\gamma'\cos B}+\ldots.=0.$$

So that the Simson line of D is

$$\frac{m-n}{la-mb\cos C-nc\cos B}\cdot\alpha+\ldots.=0.$$

The equation to Simson line which makes angles θ_1, θ_2, θ_3 with the sides of ABC is $a\cot\theta_1.\alpha+\ldots.=0$.

Hence, for the Simson line of D,

$$a\cot\theta_1 \text{ varies as } \frac{m-n}{la-mb\cos C-nc\cos B}.$$

7. To determine the radical axis of the circle on AA' as diameter, and the circle ABC.

Applying the 'power' method to the circle AA',

$$2\rho_2^2=2(BA_1^2-AA_1^2)=2(BA_1^2+AA_1^2)-AA'^2$$
$$=AB^2+A'B^2-AA'^2=2.AB.A'B.\cos B.$$

Hence, $b\rho_2^2=abc\cos B.\frac{n}{n-m}$. So $c\rho_3^2=-abc\cos C.\frac{m}{n-m}$. So that the radical axis is $n\cos B.\beta-m\cos C\gamma=0$. And the circle on AA' is

$$a\beta\gamma+b\gamma\alpha+c\alpha\beta+\frac{\lambda.abc}{4\Delta^2}(a\alpha+b\beta+c\gamma)\left(\frac{n\cos B.\beta-m\cos C.\gamma}{n-m}\right)=0.$$

Draw AE perpendicular to BC, then E lies on the circle AA'. Its coordinates are $(0,\ b\sin C\cos C,\ c\sin B\cos B)$. Hence the circle AA' is

$$a\beta\gamma+b\gamma\alpha+c\alpha\beta-(a\alpha+b\beta+c\gamma)\left(\frac{n\cos B.\beta-m\cos C.\gamma}{n-m}\right)=0.$$

It follows that the radical axis of AA', BB', is

$$\frac{n\cos B.\beta-m\cos C.\gamma}{n-m}-\frac{n\cos A.\alpha-l\,\mathrm{los}C.\gamma}{n-l}=0,$$

$$\text{or}\quad(m-n)\cos A.\alpha+(n-l)\cos B.\beta+(l-m)\cos C.\gamma=0,$$

the symmetry of the equation showing that it is the radical axis of all three circles AA', BB', CC'. This radical axis is also the line of the four orthocenters (Art. 3).

8. The radical axis of the diagonal circles is parallel to the Simson Line of D.

If $\lambda\alpha+\mu\beta+\nu\gamma=0$ makes an angle θ with BC, then

$$\cot\theta=\frac{\lambda-\mu\cos C-\nu\cos B}{\nu\sin B-\mu\sin C},$$

so that for the radical axis,

$$a\cot\theta\ \text{varies as}\ \frac{(m-n)}{la-mb\cos C-nc\cos B}.$$

Hence, the radical axis is parallel to the Simson Line (Art. 3).

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

Note on Problems 267 and 268, by W. J. GREENSTREET, M. A., Marling School, Stroud, England.

In the *Messenger of Mathematics*, 1874, Vol. III, page 137, Mr. Glaisher drew attention to the formula:

$$\tan nx=\frac{n\sin x}{\cos x+}\,\frac{(1^2-n^2)\sin^2x}{3\cos x+}\,\frac{(2^2-n^2)\sin^2x}{5\cos x+}\,\frac{(3^2-n^2)\sin^2x}{7\cos x+\ldots.}\qquad(1)$$

x being $<\frac{1}{2}\pi$, and n unrestricted.